# CHARACTERIZATION AND PROPERTIES OF EXTREME OPERATORS INTO $C(Y)$

BY

M. SHARIR

ABSTRACT

Let $E$ be a real (or complex) Banach space, $Y$ a compact Hausdorff space, and $C(Y)$ the space of real (or complex) valued continuous functions on $Y$. If $T$ is an extreme point in the unit ball of bounded linear operators from $E$ into $C(Y)$, then it is shown that $T^*$ maps (the natural imbedding in $C(Y)^*$ of) $Y$ into the weak*-closure of ext $S(E^*)$, provided that $Y$ is extremally disconnected, or $E = C(X)$, where $X$ is a dispersed compact Hausdorff space

## 1. Introduction

Suppose that $E$ is a real (or complex) Banach space. Let $S(E)$ denote its unit ball. Suppose that $Y$ is a compact Hausdorff space. We may identify $L(E, C(Y))$, the space of bounded linear operators from $E$ to $C(Y)$, with the space of weak*-continuous mappings of $Y$ into $E^*$ [5, p. 490], and the unit ball of the first space is identified with the set of those mappings which map $Y$ into the unit ball, $S(E^*)$, of $E^*$. Following Morris and Phelps [9], we call an operator $T$ in $S(L(E, C(Y)))$ a *nice* operator, if its adjoint map $T^*$ maps $Y$ into ext $S(E^*)$. It is easily verified (see [3]) that every nice operator is extreme in the unit ball of $L(E, C(Y))$. The converse assertion is not true in general. Blumenthal, Lindenstrauss and Phelps [3] have in fact, presented two examples which show that, for each compact Hausdorff space $Y$, there is a Banach space $E$ and an extreme operator $T$ in $S(L(E, C(Y)))$ such that $T^*(y)$ is extreme in $S(E^*)$ if and only if $y$ is isolated in $Y$ and if $Y = [0, 1]$, $E$ may be taken to be four-dimensional.

Consider the following two properties:

Received February 2, 1972

(A) For $T$ extreme in $S(L(E, C(Y)))$, $T^*(Y) \subset (\text{ext } S(E^*))^{-w^*}$

(B) If $E$ is a Banach space such that $\text{ext } S(E^*)$ is weak*-closed, then $T$ is extreme in $S(L(E, C(Y)))$ if and only if $T$ is a nice operator.

Neither of these properties need be true in general, as is seen by example 4 in [3]. However, it is likely that they hold for a wide class of pairs $(E, Y)$ as Theorem 4 might indicate.

A natural choice for $E$ in property $(B)$ is $C(X)$, for a compact Hausdorff space $X$, since $\text{ext } S(C(X)^*)$ is precisely $\Gamma X = \{\lambda\delta_x;\ x \in X,\ |\lambda| = 1\}$, where $\delta_x$ denotes the point measure at $x$ ([2], cf. [5, p. 441]).

The validity of (B) for $E = C(X)$ has been proved *only for* real scalars and some special $X$ or $Y$ *inspite* of the fact that it is known to be true for arbitrary compact Hausdorff spaces $X$ and $Y$ and for arbitrary scalars (real or complex) if $T$ is a *positive* linear operator, by a theorem of A. and C. Ionescu-Tulcea [7] (see also [11]). Property (A) is known to be true in one trivial case, namely, if $Y$ has a dense set of isolated points [3].

The known cases for assertion (B) with $E = C(X)$ and real scalars are:

1) $X$ is metrizable [3]
2) $X$ is Eberlein-compact and $Y$ is metrizable [1]
3) $T$ is compact [3]
4) $T$ is weakly compact and $Y$ is separable [4].

In this paper we add another two cases, both of which are true for complex scalars too.

Theorem 1 deals with a general property of extreme operators in $S(L(E, C(Y)))$, namely, the set of points in $Y$ where the adjoint map attains its maximum norm ( $= 1$) is dense in $Y$.

Theorems 2 and 4 establish (A) and (B) provided that $Y$ is extremally disconnected.

Theorem 3 is related to Kelley's theorem [8] on the characterization of $P_1$ spaces and deals with extreme extensions of extreme operators.

Theorem 5 establishes property (B) for $E = C(X)$ , where $X$ is a dispersed compact Hausdorff space, again for arbitrary scalars.

It should be noted that no selection theorems were needed in order to prove our results, while the proofs of the cases mentioned above rely mainly on their existence.

I wish to thank Dr. A. J. Lazar , under whose supervision the work on this paper was done, for his advice and constructive help.

## 2. Proofs of the theorems

For any Banach space $E$, $E^*$ will denote its dual and the action of $\mu \in E^*$ on $x \in E$ will be denoted bu $\mu(x)$. $S(E)$ will denote the closed unit ball of $E$. If $E$ and $F$ are normed linear spaces, $L(E,F)$ will denote the space of bounded linear operators from $E$ into $F$. If $E = C(X)$ for a compact Hausdorff space $X$, we may identify $E^*$ with the space of all countably-additive regular Borel measures on $X$, and we shall treat an element $\mu$ of $C(X)^*$ either as a functional or as a measure, using the same symbol. If the scalars are real, and $\mu \in C(X)^*$, $\mu^+$ and $\mu^-$ will denote the positive and the negative variations of $\mu$ respectively and one has $\mu = \mu^+ - \mu^-$. We also denote by ext $A$ the set of extreme points of $A$, where $A$ is a set in a linear space.

For each $T$ in $L(E, C(Y))$, there is a weak*-continuous mapping $T^*: Y \to E^*$ such that

$$T^*(y)(x) = Tx(y) \qquad x \in E, \qquad y \in Y$$

and

$$\| T \| = \sup \{ \| T^*(y) \| ; \qquad y \in Y\}$$

This correspondence is shown in [5. p, 490] to be one-to-one onto the space of all weak*-continuous functions from $Y$ to $E^*$. This correspondence is also linear. Hence, $T$ is extreme in $S(L(E, C(Y)))$ if and only if every weak*-continuous map $U^*: Y \to E^*$ such that $\| T^*(y) \pm U^*(y) \| \leqq 1$ for each $y \in Y$, is identically zero.

Now we are ready to prove our results.

THEOREM 1. *Let $E$ be a Banach-space, $Y$ a compact Hausdorff space, and $T$ an extreme operator in $S(L(E, C(Y)))$. Then the set $\{y \in Y; \| T^*(y) \| = 1\}$ is dense in $Y$.*

PROOF. The map $y \to \| T^*(y) \|$ is lower semi-continuous. Hence, for each integer $n$, the set

$$G_n = \{y \in Y; \| T^*(y) \| > 1 - \frac{1}{n}\}$$

is an open subset of $Y$. We shall show that $G_n$ is dense in $Y$. To this end, suppose that $f \in C(Y)$, $0 \leqq f \leqq 1$, $f | G_n \equiv 0$. Let $\mu \in S(E^*)$ be a non-zero functional; then the map $F^*: Y \to E^*$ given by $F^*(y) = (1/n) f(y)\mu$, $y \in Y$, is weak*-continuous, and we have $\| T^*(y) \pm F^*(y) \| \leqq 1$ for each $y \in Y$. Since $T$ is extreme, we must have $f \equiv 0$, and $G_n$ is therefore dense in $Y$.

Now, by Baire's category theorem,

$$\bigcap_{n=1}^{\infty} G_n = \{y \in Y;\ \| T^*(y) \| = 1\}$$

is also a dense subset of $Y$, and the proof is completed.

THEOREM 2. *Suppose that $Y$ is an extremally disconnected compact Hausdorff space and the scalars are real. Then every extreme operator in $S(L(C(X), C(Y)))$ is nice.*

PROOF. It is well-known (see [6, p. 96]) that a compact Hausdorff space $Y$ is extremally disconnected if and only if it is the Stone-Cech compactification of every dense subset.

Let $T$ be extreme in $S(L(C(X), C(Y)))$. The set $H = \{y \in Y;\ \| T^*(y) \| = 1\}$ is, by Theorem 1, a dense subset of $Y$, and therefore $Y = \beta H$ (the Stone-Cech compactification of $H$).

By using an idea from [3], we shall show now that the maps $y \to T^*(y)^+$, $y \to T^*(y)^-$, $y \in Y$, are weak*-continuous at every point of $H$. Suppose that $\{y_\alpha\}$ is a net in $Y$ converging to $y \in H$. $\{T^*(y_\alpha)^+\}$ is a bounded net in $C(X)^*$, and since the unit ball of $C(X)^*$ is weak*-compact, we may assume, by taking subnets if necessary, that $T^*(y_\alpha)^+ \xrightarrow{W^*} \mu_1 \geqq 0$ and similarly $T^*(y_\alpha)^- \xrightarrow{W^*} \mu_2 \geqq 0$. Hence

$$T^*(y_\alpha) = T^*(y_\alpha)^+ - T^*(y_\alpha)^- \to \mu_1 - \mu_2$$

But $T^*$ is weak*-continuous and so $T^*(y_\alpha) \xrightarrow{W^*} T^*(y)$. It follows that $T^*(y) = \mu_1 - \mu_2$, where $\mu_1, \mu_2 \geqq 0$. But by the minimality of the decomposition $T^*(y) = T^*(y)^+ - T^*(y)^-$, we have $\mu_1 \geqq T^*(y)^+$, $\mu_2 \geqq T^*(y)^-$. But $y \in H$ and so

$$1 = \| T^*(y) \| = \| T^*(y)^+ \| + \| T^*(y)^- \| \leqq \| \mu_1 \| + \| \mu_2 \|$$
$$\leqq \underline{\lim} \| T^*(y_\alpha)^+ \| + \underline{\lim} \| T^*(y_\alpha)^- \| \leqq \underline{\lim} \| T^*(y_\alpha) \| \leqq 1$$

Hence, every inequality is actually an equality and we must have $\mu_1 = T^*(y)^+$, $\mu_2 = T^*(y)^-$. This argument shows that the maps in question are weak*-continuous at each point of $H$, and hence weak*-continuous on $H$.

So far we have not used the fact that $Y$ is extremally disconnected, and the argument holds in general. But now, $Y = \beta H$, and $T^*(y)^+$, $T^*(y)^-$, for each $y \in H$, are in $S^+ = \{\mu \in S(C(X)^*) : \mu \geqq 0\}$, which is weak*-compact. Therefore, there exist (unique) weak*-continuous maps $T_+, T_- : Y \to S^+$, such that for each $y \in H$, $T_+(y) = T^*(y)^+$ and $T_-(y) = T^*(y)^-$, Define now another map $T : Y \to C(X)^*$ by

$$T^{\#}(y) = \| T_+(y) \| T_+(y) + \| T_-(y) \| T_-(y), \qquad y \in Y.$$

Since $\| T_{\pm}(y) \| = | T_{\pm}(y)(1) |$, $T^{\#}$ is weak*-continuous. If $y \in H$ then $\| T^*(y) \pm T^{\#}(y) \| \leqq 1$, as easily verified.

But since $T^* \pm T^{\#}$ are weak*-continuous, $H$ is dense in $Y$ and $S(C(X)^*)$ is weak*-compact, we have $\| T^*(y) \pm T^{\#}(y) \| \leqq 1$ for each $y \in Y$. Since $T$ is extreme in the unit ball of operators, we must have $T^{\#} = 0$. For $y \in H$ this means that

$$0 = T^{\#}(y)(1) = 2 \| T^*(y)^+ \| \, \| T^*(y)^- \|$$

that is, for each $y \in H$, $T^*(y) \geqq 0$ or $T^*(y) \leqq 0$. Again, using the weak*-continuity of $T^*$, this must be true for each $y \in Y$.

Now, by [3, theorem 1] this is a necessary and sufficient condition for an extreme operator $T$ to be nice, and so the proof is completed.

In order to replace $C(X)$ by a general Banach space, we make use of other properties of $C(Y)$, for an extremally disconnected compact Hausdorff space $Y$. The spaces $C(Y)$, for such $Y$, are exactly the $P_1$-spaces, namely, for any two Banach spaces $E, F$ with $E \subset F$ and any $T$ in $L(E, C(Y))$, there exists a norm-preserving extension of $T$ to $F$ (see [8]). We shall prove a finer result for *extreme* operators:

THEOREM 3. *Suppose that $Y$ is an extremally-disconnected compact Hausdorff space, $E$ and $F$ are Banach spaces and $E \subset F$. If $T_0$ is an extreme operator in $S(L(E, C(Y)))$, then there is a norm-preserving extension of $T_0$, $T: F \to C(Y)$, extreme in the unit ball of $L(F, C(Y))$.*

PROOF. Assume first that the scalars are real. With complete analogy to the standard proof of the Hahn-Banach theorem, it is easy to see (even if we confine ourselves to extreme operators) that it is sufficient to check the case where $\dim F/E = 1$.

Take $x \in F \sim E$, such that $\| x \| = 1$. By the extension property of $C(Y)$, there exists at least one norm preserving extension $T_1$ of $T_0$ to $F$. Let us define $B = \{ f \in C(Y); f = \tilde{T}x, \tilde{T}$ is a norm-preserving extension of $T_0$ to $F\}$. $B$ is a bounded family in $C(Y)$. Let $h$ be the least upper bound of $B$ in $C(Y)$, (see [6, p. 52]). If we define an extension $T_1 : F \to C(Y)$ of $T_0$ such that $T_1 x = h$, then, it is easily verified that $T_1$ is norm-preserving. $T_1$ is also extreme in $S(L(F, C(Y)))$, for suppose that

$$T_1 = \tfrac{1}{2}P + \tfrac{1}{2}Q \text{ where } P, Q \in S(L(F, C(Y))),$$

then $P|_E = Q|_E = T_0$, since $T_0$ is extreme, and

$$h = T_1 x = \tfrac{1}{2}Px + \tfrac{1}{2}Qx$$

But by the definition of $h$, $h \geqq Px$, $h \geqq Qx$, which implies that $Px = Qx = h$, whence $P = Q$ and $T_1$ must be extreme. Thus the proof for the real case is completed.

Now take the complex case, and let $T$ be any element of $L(E, C_c(Y))$, where $C_c(Y)$ denotes for the moment the space of complex-valued continuous functions on $Y$. We associate with $T$ the operator

$$T_R: E_R \to C(Y) \qquad \text{(with real scalars)}$$

such that

$$T_R^*(y) = \operatorname{Re} T^*(y) \qquad y \in Y.$$

Thus $T_R$ is a linear bounded operator from $E_R$ ($E$ considered as a real Banach space) into $C(Y)$ (the weak*-continuity of $T_R^*$ is immediate). The map $f \to \operatorname{Re} f$ between $E^*$ and $E_R^*$ (the space of real continuous linear functionals over $E$) is an isometry onto, and so $\| T_R \| = \| T \|$ and it follows easily that the map $T \to T_R$ is an affine isometry from $L(E, C_C(Y))$ onto $L(E_R, C(Y))$.

Now, let $T_0$ be extreme in $S(L(E, C_C(Y)))$. $(T_0)_R$ is then extreme in $S(L(E_R, C(Y)))$, and from the proof for the real case, it follows that there is a norm-preserving extension $T_R$ of $(T_0)_R$, extreme in $S(L(E_R, C(Y)))$. The corresponding operator $T$ has all the required properties and the proof is completed.

THEOREM 4. *Suppose that $Y$ is an extremally disconnected compact Hausdorff space and $E$ is a Banach space. If $T$ is extreme in $S(L(E, C(Y)))$ then $T^*(Y) \subset (\operatorname{ext} S(E^*))^{-w^*}$. Hence, if $E$ is a Banach space such that* ext $S(E^*)$ *is weak*-closed, then $T$ is extreme in $S(L(E, C(Y)))$ if and only if $T$ is nice.*

PROOF. Assume first the real case. Let us denote $(\operatorname{ext} S(E^*))^{-w^*}$ by $M$. $E$ can be imbedded in $C(M)$ by the mapping $x \in E \to \hat{x}|_M \in C(M)$, where $\hat{x}$ is the canonical image of $x$ in $E^{**}$. (We assign the weak*-topology to $M$, thus obtaining a compact Hausdorff space on which $\hat{x}$ is continuous.) By the Krein-Milman theorem it is easily verified that this is an isometry, so we may consider $E$ as a closed subspace of $C(M)$.

Now, by Theorem 3, $T$ may be extended to an operator $\tilde{T}$ of norm 1, such that $\tilde{T}$ is extreme in $S(L(C(M), C(Y)))$. Thus, by Theorem 2, $\tilde{T}$ is nice and we have $\tilde{T}^*(Y) \subset \pm M$. Take $y \in Y$. Then there are $\varepsilon = \pm 1$, $m \in M$ such that for each $x \in E$, $\tilde{T}^*(y)(x) = T^*(y)(x) = \varepsilon m(x)$.

But now we may consider $M$ as a subset in $E^*$, and since $M$ is symmetric we have $\varepsilon m \in M$, Hence

$$T^*(Y) \subset M = (\text{ext}\, S(E^*))^{-w^*}.$$

Now take the complex case, and use the affine isometry between $L(E, C_c(Y))$ and $L(E_R, C(Y))$ with the notations of the proof of Theorem 3. Let $T$ be extreme in $S(L(E, C_C(Y)))$ and let $T_R$ be the corresponding operator in $S(L(E_R, C(Y)))$. $T_R$ is extreme there too, and by what we have just shown $T_R^*(Y) \subset (\text{ext}\, S(E_R^*))^{-w^*}$. Using the fact that $E^*$ and $E_R^*$ are also weak*-affinely homeomorphic, it is immediate that

$$T^*(Y) \subset (\text{ext}\, S(E^*))^{-w^*}$$

Hence the proof is completed.

Corollary 1. *Let $E$ be a Banach-space such that* $\text{ext}\, S(E^*)$ *is weak*-closed, $Y$ a compact Hausdorff space and $T$ an operator in $S(L(E, C(Y)))$. Let $J$ be the canonical mapping of $C(Y)$ into $C(Y)^{**}$. Then $T$ is a nice operator if and only if $JT$ is extreme in $S(L(E, C(Y)^{**}))$.*

Proof. $C(Y)^{**}$ may be identified isometrically with $C(Y_0)$ for some extremally disconnected compact Hausdorff space $Y_0$. The operator $J: C(Y) \to C(Y_0)$ is multiplicative and therefore $J^*(Y_0) \subset Y$ (This is an easy characterization of multiplicative operators; see, e. g, [7] and [11]). Since $J$ is an isometry it is easily seen that $J^*|_{Y_0}$ must be onto $Y$. Now, assume that $T$ is a nice operator. Then

$$(JT)^*(Y_0) = T^*(J^*(Y_0)) = T^*(Y) \subset \text{ext}\, S(E^*).$$

Hence $JT$ is nice and therefore extreme in $S(L(E, C(Y)^{**}))$. Conversely, if $JT$ is extreme in $S(L(E, C(Y)^{**}))$, then, by the previous theorem, $JT$ is nice. Thus we have $T^*(Y) = (JT)^*(Y_0) \subset \text{ext}\, S(E^*)$, so that $T$ is a nice operator and the proof is completed.

Corollary 2. *The composition of two extreme operators into spaces of continuous functions, even if one of them is an isometry, need not be extreme.*

PROOF. Take $Y = [0,1]$, and $E, T$ as in [3, example 4]. Then the composition of $T: E \to C[0,1]$, with $J: C[0,1] \to C[0,1]^{**}$ is not extreme, by the previous corollary.

We now establish property (B) when $E = C(X)$, where $X$ is a dispersed compact Hausdorff space.

Let $X, Y$ be any compact Hausdorff spaces and $T: C(X) \to C(Y)$ a bounded linear operator. For each $x \in X$ we define a scalar function $T_x$ on $Y$ by

$$T_x(y) = T^*(y)(\{x\}), \qquad y \in Y$$

LEMMA. *Let $T$ be an extreme operator in $S(L(C(X), C(Y)))$, $V \subset Y$ a non-void open subset and $x_1 \neq x_2$ in $X$ such that $T_{x_1}, T_{x_2}$ are continuous on $V$. Then for each $y \in V$*

$$T_{x_1}(y) T_{x_2}(y) = 0$$

PROOF. Suppose, on the contrary, that there is $y_0$ in $V$ such that $T_{x_1}(y_0)$ $T_{x_2}(y_0) \neq 0$. Define

$$V_1 = \{y \in V;\ |T_{x_1}(y)| \geqq |T_{x_2}(y)|\}$$
$$V_2 = \{y \in V;\ |T_{x_1}(y)| \leqq |T_{x_2}(y)|\}$$

then $V_1$, $V_2$ are relatively closed in $V$ and $V_1 \cup V_2 = V$. Define a map $g: V \to C(X)^*$ by

$$g(y) = \begin{cases} w[T_{x_1}(y)]\,|T_{x_2}(y)|\,\delta_{x_1} - T_{x_2}(y)\delta_{x_2} & y \in V_1, \\ T_{x_1}(y)\delta_{x_1} - w[T_{x_2}(y)]\,|T_{x_1}(y)|\,\delta_{x_2}, & y \in V_2 \end{cases}$$

where

$$w(z) = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Since on $V_1 \cap V_2$ the two definitions coincide, $g$ is weak*-continuous on $V$. Now for each $y \in Y$ we have the decomposition $T^*(y) = \mu_y + T_{x_1}(y)\delta_{x_1} + T_{x_2}(y)\delta_{x_2}$ where

$$\mu_y = T^*(y)|_{X \sim \{x_1, x_2\}}$$

and

$$\|T^*(y)\| = \|\mu_y\| + |T_{x_1}(y)| + |T_{x_2}(y)|$$

Then we have for each $y \in V$, $\| T^*(y) \pm g(y) \| \leqq 1$.

Now, by our initial assumption, $g(y_0) \neq 0$. Take $h \in C(Y)$ such that $0 \leqq h \leqq 1$, $h(y_0) = 1$, and $h|_{Y \setminus V} \equiv 0$, and define $\tilde{g}\colon Y \to C(X)^*$ by

$$\tilde{g}(y) = \begin{cases} h(y)g(y), & y \in V \\ 0 \quad , & y \notin V. \end{cases}$$

It follows that $\tilde{g}$ is weak*-continuous and that for each $y \in Y$, $\| T^*(y) \pm \tilde{g}(y) \| \leqq 1$. But this is a contradiction of the fact that $T$ is extreme in the unit ball of operators, for $\tilde{g}(y_0) = h(y_0)g(y_0) \neq 0$, and so the Lemma is proved.

We recall that a topological space $X$ is called *dispersed* if it contains no non-void perfect subsets. If $X$ is a compact Hausdoff space, then, by a theorem of Rudin [12] (cf. [10]), $X$ is dispersed only if $C(X)^*$ is $l^1(X)$, namely, each $\mu$ in $C(X)^*$ admits a representation $\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n}$ with $\| \mu \| = \sum_{n=1}^{\infty} |a_n| < \infty$ and $x_n \in X$ for each integer $n$.

THEOREM 5. *Let $X$ be a dispersed compact Hausdorff space, $Y$ a compact Hausdorff space and $T$ an extreme operator in $S(L(C(X), C(Y)))$. Then $T$ is a nice operator.*

PROOF. Define $V = \{y \in Y;\ T^*(y) \notin \Gamma X\}$. Since $\Gamma X$ is weak*-closed, $V$ is an open subset of $Y$. We want to show that $V$ is void, so let us assume, on the contrary, that $V \neq \varnothing$.

Let $\beta$ be an ordinal and $L_\beta$ the $\beta$-th derived set of $X$ (where we put $L_0 = X$). Let $I_\beta$ be the set of (relatively) isolated points of $L_\beta$. Since $X$ is dispersed, then $I_\beta$ is non-void when $L_\beta$ is non-void. Let $G_\beta = \cup \{I_\gamma;\ \gamma < \beta\}$, then it is clear that $G_\beta \cup L_\beta = X$ and $G_\beta \cap L_\beta = \varnothing$,

Let $\alpha$ be the smallest ordinal such that there exists $x \in I_\alpha$ for which $T_x$ does not vanish identically on $V$. There has to be such an ordinal, since otherwise, for each $x \in X$, $T_x|_V \equiv 0$, and since $C(X)^*$ contains only purely-atomic measures, we must have $T^*|_V \equiv 0$ which is a contradiction by Theorem 1.

We shall show that for each $x \in I_\alpha$, $T_x$ is continuous on $V$. Indeed, let $x \in I_\alpha$. Since $x$ is isolated in $L_\alpha$, $U = G_\alpha \cup \{x\}$ is an open neighbourhood of $x$. Let $f \in C(X)$ such that $f_{X \setminus U} \equiv 0$ and $f(x) = 1$. Then the mapping $y \to T^*(y)(f)$ is continuous on $Y$, but for each $y \in V$, $T^*(y)(f) = T_x(y)$. Indeed, one has $T^*(y)(f) = \sum_{n=1}^{\infty} a_n f(x_n)$ for a suitable sequence $\{a_n\}$ of non-zero scalars and a sequence $\{x_n\}$ in $X$. But, by the definition of $\alpha$, each $x_n$ is either $x$ or is in $X \setminus U$. and the result follows easily. Thus $T_x$ is continuous on $V$ for each $x \in I_\alpha$.

Now we may define

$$V_0 = \{y \in V;\ T_x(y) \neq 0 \text{ for some } x \in I_\alpha\}\ .$$

Since the $T_x$ are continuous functions on $V$ for $x \in I_\alpha$, $V_0$ is a non-void open subset of $Y$. Using the same method as before, it is easy to prove by transfinite induction the following two claims for an ordinal $\beta \geqq \alpha$:

I. $T_x$ is continuous on $V_0$ for each $x \in I_\beta$.

II. For each $y \in V_0$, $T_x(y) \neq 0$ for at most one $x \in \{I_\gamma\,;\ \gamma \leqq \beta\}$.

Claim II follows directly from the preceding Lemma, once claim I is established. Hence, the claims are true for all $x \in X$.

Let us now apply Theorem 1. Define

$$H = \{y \in Y;\ \| T^*(y) \| = 1\},$$

$H$ is dense in $Y$ and so there is $y_0 \in H \cap V_0$. We have a representation $T^*(y_0) = \sum_{n=1}^{\infty} a_n \delta_{x_n}$ where $\sum_{n=1}^{\infty} | a_n | = 1$ and $\{x_n\} \subset X$. Now, for each integer $n$, $T_{x_n}(y_0) = a_n$, and by claim II, $a_n \neq 0$ for at most one integer $n$, say $n_0$. Hence $T^*(y_0) = a_{n_0}\delta_{x_{n_0}}$ and $| a_{n_0} | = 1$. Whence $T^*(y_0) \in \Gamma X$, which contradicts the definition of $V_0$. Thus $V$ is void and $T$ is therefore a nice operator.

## References


1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach Spaces*, Ann. of Math. **88** (1968), 35–46.

2. R. F. Arens and J. L. Kelley, *Characterization of the space of continuous functions over a compact Hausdorff space*, Trans. Amer. Math. Soc. **62** (1947), 499–508.

3. R. M. Blumenthal, J. Lindenstrauss and R. R. Phelps, *Extreme operators into C(K)*, Pacific J. Math. **15** (1965), 747–756.

4. H. Corson and J. Lindenstrauss, *Continuous selections with nonmetrizable range*, Trans. Amer. Math. Soc. **121** (1966), 492–504.

5. N. Dunford and J. Schwartz, *Linear Operators, Part I : General Theory*, Interscience, New York, 1958.

6. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

7. A. and C. Ionescu Tulcea, *A note on extreme points*, Unpublished.

8. J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326.

9. P. D. Morris and R. R. Phelps, *Theorems of Krein-Millman type for certain convex sets of operators*, Trans. Amer. Math. Soc. **150** (1970), 183–200.

10. A. Pelczynski and Z. Semadeni, *Spaces of continuous functions, III: Spaces C(Ω) for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222.

11. R. R. Phelps, *Extreme positive operators and homomorphisms*, Trans. Amer. Math. Soc. **108** (1963), 265–274.

12. W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42.



Department of Mathematical Sciences
Tel Aviv University, Tel Aviv